

Quadrature entanglement and photon-number correlations accompanied by phase-locking

H. H. Adamyan,^{1,*} N. H. Adamyan,^{2,†} S. B. Manvelyan,^{1,‡} and G. Yu. Kryuchkyan^{2,1,§}

¹*Institute for Physical Research, National Academy of Sciences,
Ashtarak-2, 378410, Armenia*

²*Yerevan State University, A. Manookyan 1, 375049, Yerevan, Armenia*

We investigate quantum properties of phase-locked light beams generated in a nondegenerate optical parametric oscillator (NOPO) with an intracavity waveplate. This investigation continues our previous analysis presented in Phys.Rev.A **69**, 05814 (2004), and involves problems of continuous-variable quadrature entanglement in the spectral domain, photon-number correlations as well as the signatures of phase-locking in the Wigner function. We study the role of phase-localizing processes on the quantum correlation effects. The peculiarities of phase-locked NOPO in the self-pulsing instability operational regime are also cleared up. The results are obtained in both the P -representation as a quantum-mechanical calculation in the framework of stochastic equations of motion, and also by using numerical simulation based on the method of quantum state diffusion.

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I. INTRODUCTION

In the presently very active field of continuous variable (CV) quantum information processing a challenging goal consists in generation of entangled intensive light beams. Various quantum optical schemes generating entangled bright light have been proposed for this goal. For the first time the CV entangled states of light have been studied in [1] and demonstrated experimentally in [2] for nondegenerate optical parametric oscillator (NOPO) below the threshold. An experimental progress in this direction has been recently reported in [3, 4]. However, up to now the generation of bright light beams with high level of CV entanglement meets serious problems. One of these is degradation of entanglement due to uncontrolled dissipation and decoherence, and cavity induced feedback. Beside this, it is known, that the relative phase between subharmonics in above threshold NOPO undergoes diffusion process. This process destroys the frequency degeneracy of modes and hence limits the production of quantum-twin beams in NOPO above threshold. For reducing such phase diffusion phenomena various methods, based on phase locking mechanisms have been proposed [5, 6, 7, 8, 9, 10].

The simplest scheme proposed and experimentally realized by Mason and Wong [5] is the NOPO with additional intracavity quarter-wave plate to provide polarization mixing between two orthogonally polarized modes of the subharmonics. Indeed, it was shown that induced linear coupling between generated modes results in a locking phenomenon and hence in frequency degenerate operation above threshold. Following this experiment, the

semiclassical theory of NOPO with self-phase locking was developed in [6] and its further more detailed consideration has been presented by Fabre and colleagues [11]. In a more general approach the semiclassical theory of the self-phase-locked NOPO has been investigated by Groβ and Boller [7].

Type-II optical parametric oscillators with self-phase locking have recently attracted a lot of attention as efficient source of nonclassical light [12, 13, 14, 15, 16, 17]. A full quantum mechanical treatment of this system in application to generation of CV entangled states of light beams under mode phase-locked condition has been presented in [12, 13], where the regimes below, near and above threshold was considered. Quantum optical effects have been demonstrated by Fabre group in the series of experiments [14, 15, 16]. Generation of quadrature EPR entanglement has been demonstrated experimentally [14] with self-phase-locked NOPO below threshold. Experimental investigation of intensity quantum correlation with phase-locked NOPO above threshold has been also performed [15]. The intensity-difference squeezing in electronically phase locked NOPO above threshold as well as the Hong-Ou-Mandel interferometry using twin beams have also been experimentally demonstrated by Feng and Pfister [17]. Stable generation in application to metrology has been also demonstrated [18].

In this paper we continue investigation of phase-locked NOPO follow the paper [12]. The previous studies of entanglement in the presence of phase localizing processes have only concentrated on stable stationary solutions for dynamics of intracavity subharmonics. Quite recently, the experimental observation of dynamical signatures of self-phase locking in a triply resonant degenerate OPO was reported in [19] for both stable and unstable regimes, however, the study of dynamical aspects of entanglement for this system have been postponed for future investigations. In addition, here we will consider entanglement production in the unstable regime of generation. We shall also give below the quantum description of the unstable

*adam@unicad.am

†Narek.Adamyan@synopsys.com

‡smanvel@server.physdep.r.am

§gkryuchk@server.physdep.r.am

regime in phase-space on the framework of Wigner function.

Our goal is two-fold. In one part of the present paper, we expand the previous study of phase-locked NOPO in the stable, steady-state regime of generation [12], particularly considering both the important details in phase-space in the frame of Wigner function and photon-numbers difference squeezing in the presence of phase locking. We also investigate quadrature EPR entanglement in the spectral domain in addition to our previous results which have been performed in the time domain. The other part of the paper is devoted to the problem of entanglement in the self-pulsing instability regime.

The system under consideration is the NOPO with quarter wave plate inside the cavity, interacting with the thermal bath. Due to explicit presence of dissipation in this problem, one has to write the master equation for the reduced density matrix of the system, which within the framework of the rotating wave approximation and in the interaction picture is

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} [H, \rho] + \sum_{i=1}^3 \gamma_i (2a_i \rho a_i^\dagger - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i), \quad (1)$$

where

$$H = \sum_{i=1}^3 \hbar \Delta_i a_i^\dagger a_i + i\hbar E (a_3^\dagger - a_3) + i\hbar k (a_3 a_1^\dagger a_2^\dagger - a_3^\dagger a_1 a_2) + \hbar \chi (a_1^\dagger a_2 + a_1 a_2^\dagger), \quad (2)$$

where a_i are the boson operators for the cavity modes ω_i . The mode a_3 at frequency ω is driven by an external field with amplitude E , while a_1 and a_2 describe subharmonics of two orthogonal polarizations at the degenerate frequency $\omega/2$. The constant k determines the efficiency of the down-conversion process, while χ describes the energy exchange between only the subharmonic modes due to the intracavity waveplate. It should be noted that maximally reached value for χ in the experiments is $\chi/\gamma \simeq 0.21$. However in the following calculations we consider χ/γ in a more wide range keeping in mind future experimental achievements. We take into account the detunings of subharmonics Δ_i and the cavity damping rates γ_i and consider the case of high cavity losses for pump mode ($\gamma_3 \gg \gamma_1, \gamma_2$), when this mode can be adiabatically eliminated. However, in our analysis we take into account the pump depletion effects. We will solve Eq.(1) in the framework of P -representation and stochastic variables on one side, and also by using the well known numerical quantum state diffusion method (QSD), according to which, open quantum systems are represented by the ensemble of quantum trajectories [20]. The density matrix is restored in this case from the ensemble averaging over photonic Fock states.

The paper is arranged as follows. In Sec. II we recall some theoretical points of self-phase locked NOPO

and we present analysis of both photon-number quantum correlation and phase locking in the base of the Wigner functions of generated modes. Section III is devoted to an analysis of quantum fluctuations of both modes in below-threshold operational regime as well as to calculation of the squeezed quadrature variance. In Sec. IV we investigate the self-pulsing instability regime of NOPO on the base of quantum trajectories and in the phase space. We also discuss there the CV entanglement in the unstable regime of generation. We summarize our results in Sec.V.

II. SIGNATURES OF PHASE-LOCKING IN THE QUANTUM CORRELATIONS AND IN THE PHASE SPACE

At first, we shortly discuss the phase-locking phenomena in the semiclassical theory of NOPO. The stochastic equations of self-locked NOPO for the complex c-number variables α_i and β_i corresponding to the operators a_i and a_i^\dagger , in the regime of adiabatic elimination of pump mode have the following form [12]:

$$\begin{aligned} \frac{\partial \alpha_1}{\partial t} = & -(\gamma_1 + i\Delta_1) \alpha_1 \\ & + (\varepsilon - \lambda \alpha_1 \alpha_2) \beta_2 - i\chi \alpha_2 + R_1, \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial \beta_1}{\partial t} = & -(\gamma_1 - i\Delta_1) \beta_1 \\ & + (\varepsilon - \lambda \beta_1 \beta_2) \alpha_2 + i\chi \beta_2 + R_1^\dagger. \end{aligned} \quad (4)$$

Here $\varepsilon = kE/\gamma_3$ and $\lambda = k^2/\gamma_3$. The equations for α_2 and β_2 are obtained by changing the indexes (1) \leftrightarrow (2). R_1, R_2 are Gaussian noise terms obeying the following correlations

$$\langle R_1(t) R_2(t') \rangle = \frac{k}{\gamma_3} (E - k \alpha_1 \alpha_2) \delta(t - t'), \quad (5)$$

$$\langle R_1^\dagger(t) R_2^\dagger(t') \rangle = \frac{k}{\gamma_3} (E - k \beta_1 \beta_2) \delta(t - t'). \quad (6)$$

It has been shown in [11, 12] that these equations in the semiclassical approximation have stationary stable solutions only if the following relation holds

$$4\chi^2 \Delta_1 \Delta_2 > (\gamma_1 \Delta_2 - \gamma_2 \Delta_1)^2, \quad (7)$$

otherwise the system has only unstable solutions.

It should be noted that the phase locking condition (7) obtained here in the adiabatic approximation of triply resonant NOPO has a universal form and is realized for the various schemes of self-phase-locked NOPO. It has been obtained at first [5, 6] for the case of doubly resonant NOPO. As shown in [7] this condition coincides with analogous condition obtained for triply resonant NOPO without adiabatic approximation.

Let us recall the results concerned to the phases and photon numbers for the stationary, stable regime of self locked NOPO for the cases of $\Delta_1 = \Delta_2 \equiv \Delta$, and $\gamma_1 = \gamma_2 = \gamma$ [12]. In above threshold regime $\varepsilon \geq \varepsilon_{th} = \sqrt{(\chi - |\Delta|)^2 + \gamma^2}$, the steady-state solution for the photon numbers is:

$$n_0 = n_{10} = n_{20} = \frac{1}{\lambda} \left[\sqrt{\varepsilon^2 - (\chi - |\Delta|)^2} - \gamma \right], \quad (8)$$

while the phases have been found to be

$$\varphi_{10} = \varphi_{20} = -\frac{1}{2} \text{Arcsin} \frac{1}{\varepsilon} (\chi + |\Delta|) + \pi k, \quad (9)$$

for $\Delta > 0$. For the opposite sign of the detuning, $\Delta < 0$ the mean photon numbers are given by the same Eq.(8), while the phases read as

$$\varphi_{10} = \frac{1}{2} \text{Arcsin} \frac{1}{\varepsilon} (\chi + |\Delta|) + \pi \left(k + \frac{1}{2} \right), \quad (10)$$

$$\varphi_{20} = \frac{1}{2} \text{Arcsin} \frac{1}{\varepsilon} (\chi + |\Delta|) + \pi \left(k - \frac{1}{2} \right),$$

($k = 0, 1, 2, \dots$). Analyzing the system with help of a linear treatment of quantum fluctuations in the P -representation we have arrived [12] to the following correlators

$$\left\langle \begin{pmatrix} \delta n_+ \\ \delta \varphi_+ \end{pmatrix} \begin{pmatrix} \delta n_+, \delta \varphi_+ \end{pmatrix} \right\rangle = \frac{1}{4\lambda n_0(\gamma + \lambda n_0)} \begin{pmatrix} 4n_0 [\gamma(\gamma + \lambda n_0) + (\chi - |\Delta|)^2], & -\lambda n_0(\chi - |\Delta|) \text{sign}(\Delta) \\ -2\lambda n_0(\chi - |\Delta|) \text{sign}(\Delta), & -\lambda \gamma \end{pmatrix}, \quad (11)$$

$$\left\langle \begin{pmatrix} \delta n_- \\ \delta \varphi_- \end{pmatrix} \begin{pmatrix} \delta n_-, \delta \varphi_- \end{pmatrix} \right\rangle = \frac{1}{4|\Delta|\chi} \begin{pmatrix} 4n_0\chi(\chi - |\Delta|), & 2\chi\gamma \text{sign}(\Delta) \\ 2\chi\gamma \text{sign}(\Delta), & \frac{1}{n_0}(\gamma^2 - |\Delta|(\chi - |\Delta|)) \end{pmatrix}, \quad (12)$$

where $\delta n_{\pm} = \delta n_2 \pm \delta n_1$, $\delta \varphi_{\pm} = \delta \varphi_2 \pm \delta \varphi_1$, and $\delta n_i(t) = n_i(t) - n_{i0}$ and $\delta \varphi_i(t) = \varphi_i(t) - \varphi_{i0}$. These correlators is written in terms of the stochastic variables, namely between photon-number sum and phase sum in the modes, as well as between photon-number difference and phase difference in the modes. What is now interesting for us is the dependence of both photon number quantum correlation and phase fluctuations on the wave plate parameter χ .

A. Photon-number quantum correlations in the presence of phase-locking

Let us at first consider the quantum correlations in twin light beams generated in self-phase locked NOPO. Intensity correlations of twin light beams are usually characterized quantitatively by the quantum fluctuations of the intensity difference between the generated beams which is normalized to the corresponding shot noise level. In this way the intense twin beams quantum correlations have been experimentally observed several years ago in NOPO operated above its threshold [21, 22]. Up to now the several interesting applications of intensity correlation have been proposed and realized [23, 24, 25]. In this section we analyze shortly the number-difference squeezing in the presence of phase localizing process. Most published studies of quantum correlations have usually been performed in the spectral domain by measure of the corresponding squeezing spectra. However, here we restrict ourselves by calculation only the integral quan-

tity. To characterize the photon-number correlation we address to the variance of the fluctuations in the photon number difference:

$$R = \langle (a_1^\dagger a_1 - a_2^\dagger a_2)^2 \rangle - (\langle n_1 \rangle - \langle n_2 \rangle)^2. \quad (13)$$

This variance is expressed through the stochastic variables using the relationship between normally-ordered moments of the operators and the stochastic moments with respect to the P -function. In the balance case $\langle n_1 \rangle = \langle n_2 \rangle = \langle n \rangle$ and in the framework of the stochastic variables we get

$$R = 2\langle n \rangle + \langle \delta n_-^2 \rangle. \quad (14)$$

Then, using the formulas (12) we obtain

$$R = n_0 \frac{|\Delta| + \chi}{|\Delta|}. \quad (15)$$

Thus, the variance normalized to the level of fluctuations for the coherent state is

$$R_N = \frac{R}{2n_0} = \frac{1}{2} \left(1 + \frac{\chi}{|\Delta|} \right). \quad (16)$$

For $\chi \rightarrow 0$, $R_N = 1/2$, i.e. the normalized variance reaches 50 percent relative to the quantum noise level of a coherent state, in agreement with the result of the paper [26] devoted to an ordinary NOPO. As we see,

for $\chi/|\Delta| \ll 1$ there is a small aggravation of the quantum correlation between the photon numbers of the corresponding twin fields in a self-phase locked NOPO in comparison with the case of an ordinary NOPO.

The effect arising from the photon correlation in each modes as well as between twin modes are usually investigated with the aid of the second-order correlation functions

$$g_{ii} = \langle a_i^{\dagger 2} a_i^2 \rangle, \quad g_{12} = \langle a_1^{\dagger} a_2^{\dagger} a_1 a_2 \rangle, \quad (17)$$

where $(i = 1, 2)$. We calculate these correlations in the linear treatment of quantum fluctuations and in terms of the stochastic variables as

$$g_{11} = g_{22} = g = n_0^2 + \frac{1}{4}(\langle \delta n_+^2 \rangle + \langle \delta n_-^2 \rangle), \quad (18)$$

$$g_{12} = n_0^2 + \frac{1}{4}(\langle \delta n_+^2 \rangle - \langle \delta n_-^2 \rangle).$$

It is easy to verify that the following relation between the correlation functions hold

$$g_{12} = g - \frac{1}{2}\langle \delta n_-^2 \rangle = g + \frac{n_0}{2}\left(1 - \frac{\chi}{|\Delta|}\right), \quad (19)$$

while the correlator g equals to

$$g = n_0^2 + \frac{\gamma}{4\lambda}\left(1 + \frac{(\chi - |\Delta|)^2}{\gamma(\gamma + \lambda n_0)}\right) - \frac{n_0}{4}\left(1 - \frac{\chi}{|\Delta|}\right). \quad (20)$$

When the wave plate is not included, the formula (19) is transformed to the relation between the correlation functions of an ordinary NOPO [26].

As we see, the insertion of a polarization mixer slightly destroys the photon number correlations produced by an ordinary NOPO. Having discussed the influence of phase locking on photon-number quantum correlations, we now turn our attention to investigate the Wigner function of self-phase locked NOPO.

B. Phase locking on the Wigner function

It is clearly seen from the Eqs. (11) and Eqs.(12) that for sufficiently small χ the terms $\langle \delta n_+^2 \rangle$, $\langle \delta n_-^2 \rangle$ and $\langle \delta \varphi_+^2 \rangle$ does not depend on χ , while the term $\langle \delta \varphi_-^2 \rangle$ is inversely proportional to it. Indeed, from Eqs.(12) we have

$$\langle \delta \varphi_-^2 \rangle = \frac{1}{4|\Delta|\chi n_0}[\gamma^2 - |\Delta|(\chi - |\Delta|)]. \quad (21)$$

Therefore, decreasing of χ in the correlator leads to increasing of phase difference fluctuations, and at last with $\chi = 0$ we arrive to well known fact of phase diffusion in

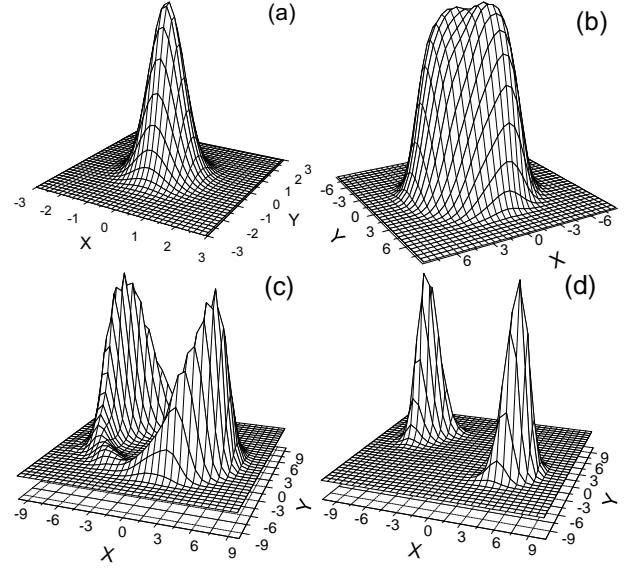


FIG. 1: The Wigner function of the self-phase locked NOPO in the stable regime for the parameters: $\lambda/\gamma = 0.1$, $\Delta_1/\gamma = \Delta_2/\gamma = 10$, $\chi/\gamma = 0.1$, a) $\varepsilon/\gamma = 5$, (below threshold), b) $\varepsilon/\gamma = 10$, (at threshold), c) $\varepsilon/\gamma = 11$, (above threshold), d) $\varepsilon/\gamma = 11$, but $\chi/\gamma = 0.5$. Averaging is over 3000 quantum trajectories.

NOPO without wave plate. This phenomenon also exhibits himself in the framework of Wigner function and we turn to the consideration of this point.

It has been found in [12] from the general point of view that the subharmonic modes has the phase space symmetry properties: the Wigner functions W_1 and W_2 of the modes have a two-fold symmetry under the rotation of the phase-space by angle π around its origin,

$$W_i(r, \theta) = W_i(r, \theta + \pi). \quad (22)$$

Here r, θ are the polar coordinates of the complex phase space. We show below the results of numerical simulations in the framework of QSD method. Figure 1 plots the Wigner function of one on the subharmonics of self-phase locked NOPO over transient time and for all operational regimes. A clear self-phase locking in the transition through the generation threshold is seen. The Wigner function is almost Gaussian below threshold and is squeezed near the threshold. Above-threshold the Wigner function has two separated peaks due to the phase-localizing processes leading to the phase locking of the subharmonic modes. The distance between peaks increases with increasing the photon number (decreasing λ). Note, that the decreasing of wave plate parameter χ leads to less localization of these peaks (compare Fig.1(d) ($\chi/\gamma = 0.5$) with Fig.1(c) ($\chi/\gamma = 0.1$)).

III. QUADRATURE ENTANGLEMENT IN THE SPECTRAL DOMAIN

In this section we continue investigation of CV entanglement in the self-phase locked NOPO. In the previous studies [12] we have calculated the variances of the relevant distance $V_- = V(X_1 - X_2)$ and the total momentum $V_+ = V(Y_1 + Y_2)$ of the quadrature amplitudes of two modes $X_k = \frac{1}{\sqrt{2}} [a_k^+ \exp(-i\theta_k) + a_k \exp(i\theta_k)]$, $Y_k = \frac{i}{\sqrt{2}} [a_k^+ \exp(-i\theta_k) - a_k \exp(i\theta_k)]$, ($k = 1, 2$), where $V(X) = \langle X^2 \rangle - \langle X \rangle^2$ is a denotation for the variance and θ_k is the phase of local oscillator for the k -th mode. The two quadratures X_k and Y_k are non commuting observables. We have considered the variance in the time-domain and have quantified the CV entanglement by the inseparability criterion for the quantum state of two optical modes [27]

$$V = \frac{1}{2}(V_+ + V_-) < 1. \quad (23)$$

It has been demonstrated recently that time-dependent quadrature variance could be observed by means of time-resolved homodyne measurements [28]. This approach particularly allows for applications in time-resolved quantum information protocols. Nevertheless, so far squeezing as well as CV entanglement have been mainly demonstrated in the spectral domain and not in the time domain. Therefore, we calculate here the variance V in the spectral domain and in a fully quantum mechanical treatment of self-phase locked NOPO below threshold, $E < E_{th}$.

We use the stochastic equations (3), (4) linearized around the zero-amplitude solution $\alpha_i^0 = \beta_i^0 = 0$ in the following matrix form for small fluctuations $\alpha_i(t) =$

$\delta\alpha_i(t)$ and $\beta_i(t) = \delta\beta_i(t)$, ($i = 1, 2$)

$$\frac{d\vec{L}}{dt} = -\hat{F}\vec{L} + \vec{R}. \quad (24)$$

Here

$$\hat{F} = \begin{pmatrix} \hat{A} & -\hat{B} \\ -\hat{B}^* & \hat{A}^* \end{pmatrix}, \vec{R}(t) = \begin{pmatrix} \vec{R}_\alpha(t) \\ \vec{R}_\beta(t) \end{pmatrix}, \vec{L}(t) = \begin{pmatrix} \delta\vec{\alpha}(t) \\ \delta\vec{\beta}(t) \end{pmatrix} \quad (25)$$

where $\delta\vec{\alpha} = (\delta\alpha_1, \delta\alpha_2)^T$, $\delta\vec{\beta} = (\delta\beta_1, \delta\beta_2)^T$, $\vec{R}_\alpha = (R_1, R_2)^T$, $\vec{R}_\beta = (R_1^+, R_2^+)^T$ are two-dimensional column vectors. The 4×4 matrix \hat{F} is written in the block form with 2×2 matrices

$$\hat{A} = \begin{pmatrix} \gamma + i\Delta & i\chi \\ i\chi & \gamma + i\Delta \end{pmatrix}, \hat{B} = \varepsilon\hat{\sigma}, \hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (26)$$

The noise correlators are determined as

$$\langle \vec{R}_\alpha(t) \vec{R}_\alpha^T(t') \rangle = \hat{D} \delta(t - t') \quad (27)$$

with the following diffusion matrix:

$$\hat{D} = \begin{pmatrix} \hat{B} & 0 \\ 0 & \hat{B}^* \end{pmatrix}. \quad (28)$$

The correlation functions of the quantum fluctuations are obtained in the following form (we show results of [12] with changing some misprints)

$$\begin{aligned} \langle \delta\alpha(\delta\alpha)^T \rangle &= \frac{\varepsilon}{2(S^4 - 4\Delta^2\chi^2)} \left[\gamma \begin{pmatrix} -2\chi\Delta & S^2 \\ S^2 & -2\chi\Delta \end{pmatrix} - i \begin{pmatrix} \chi(S^2 - 2\Delta^2) & \Delta(S^2 - 2\chi^2) \\ \Delta(S^2 - 2\chi^2) & \chi(S^2 - 2\Delta^2) \end{pmatrix} \right], \\ \langle \delta\alpha(\delta\beta)^T \rangle &= \frac{\varepsilon^2}{2(S^4 - 4\Delta^2\chi^2)} \begin{pmatrix} S^2 & -2\chi\Delta \\ -2\chi\Delta & S^2 \end{pmatrix}, \end{aligned} \quad (29)$$

where S^2 is introduced as

$$S^2 = \gamma^2 + \chi^2 + \Delta^2 - \varepsilon^2. \quad (30)$$

These correlators are obtained in a steady state regime

and do not dependent on time. However, for spectrally resolved measurements of the cavity output fields we need in the correlation functions in the spectral domain. To calculate them we use the Fourier-transformed linearized equations for the stochastic amplitudes

$$\vec{\alpha}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \vec{\alpha}(\omega) d\omega, \quad \vec{\alpha}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \vec{\alpha}(t) dt, \quad (31)$$

$$\vec{\beta}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \vec{\beta}(\omega) d\omega, \quad \vec{\beta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \vec{\beta}(t) dt, \quad (32)$$

which are obtained as given below

$$\begin{cases} i\omega \delta \vec{\alpha}(\omega) = -\hat{A} \delta \vec{\alpha}(\omega) + \hat{B} \delta \vec{\beta}(\omega) + \vec{R}_\alpha(\omega), \\ i\omega \delta \vec{\beta}(\omega) = \hat{B}^* \delta \vec{\alpha}(\omega) - \hat{A}^* \delta \vec{\beta}(\omega) + \vec{R}_\beta(\omega). \end{cases} \quad (33)$$

Here, the nonzero correlation functions of noise terms in the frequency space are

$$\langle \vec{R}(\omega) \vec{R}^T(\omega') \rangle = \hat{D} \delta(\omega + \omega'). \quad (34)$$

We rewrite these equations in the more compact form for

$$\vec{L}(\omega) = \begin{pmatrix} \delta \vec{\alpha}(\omega) \\ \delta \vec{\beta}(\omega) \end{pmatrix} \text{ as}$$

$$\vec{L}(\omega) = (\hat{F} + i\omega \hat{I})^{-1} \vec{R}(\omega) \quad (35)$$

and then using the correlators from Eq.(34), we explicitly calculate

$$\begin{aligned} \langle \vec{L}(\omega) \vec{L}^T(\omega') \rangle &= \langle (\hat{F} + i\omega \hat{I})^{-1} \vec{R}(\omega) \vec{R}^T(\omega') (\hat{F} + i\omega' \hat{I})^{-1T} \rangle = \\ &= (\hat{F} + i\omega \hat{I})^{-1} \hat{D} (\hat{F} + i\omega' \hat{I})^{-1T} \delta(\omega + \omega'). \end{aligned} \quad (36)$$

This result can be transformed to the following form

$$\begin{aligned} \langle \vec{L}(\omega) \vec{L}^T(\omega') \rangle &= (\hat{F} + i\omega \hat{I})^{-1} (\hat{F} + i\omega' \hat{I})^{-1} \hat{D} \delta(\omega + \omega') = \\ &= (\hat{F}^2 + \omega^2 \hat{I})^{-1} \hat{D} \delta(\omega + \omega') \end{aligned} \quad (37)$$

by using the formula

$$\hat{D}(\hat{F} + i\omega \hat{I})^{-1T} = (\hat{F} + i\omega \hat{I})^{-1} \hat{D}. \quad (38)$$

The further simplification of this result is connected with the structures of the matrices \hat{F} , \hat{D} which are written in the block forms (25),(28). As a consequence we have

$$\hat{F}^2 + \hat{I}\omega^2 = \begin{pmatrix} a\hat{I} + b\hat{\sigma} & -c\hat{\sigma} \\ -c^*\hat{\sigma} & a^*\hat{I} + b^*\hat{\sigma} \end{pmatrix}, \quad (39)$$

where I is the identity matrix and the following notations are used

$$\begin{aligned} a &= (\gamma + i\Delta)^2 - \chi^2 + \varepsilon^2 + \omega^2, \\ b &= 2i\chi(\gamma + i\Delta), \\ c &= 2\gamma\varepsilon. \end{aligned} \quad (40)$$

As a result, the correlation matrices (37) can be also written in the following block form

$$\langle \vec{L}(\omega) \vec{L}^T(\omega') \rangle = \begin{pmatrix} \hat{G}_0 & \hat{G}_1 \\ \hat{G}_1^* & \hat{G}_0^* \end{pmatrix} \delta(\omega + \omega'), \quad (41)$$

where the matrices \hat{G}_0 , \hat{G}_1 can be calculated from the equations

$$\begin{pmatrix} a\hat{I} + b\hat{\sigma} & -c\hat{\sigma} \\ -c^*\hat{\sigma} & a^*\hat{I} + b^*\hat{\sigma} \end{pmatrix} \begin{pmatrix} \hat{G}_0 & \hat{G}_1 \\ \hat{G}_1^* & \hat{G}_0^* \end{pmatrix} = \begin{pmatrix} \varepsilon\hat{\sigma} & 0 \\ 0 & \varepsilon\hat{\sigma} \end{pmatrix}, \quad (42)$$

which can be transformed to the following form:

$$\begin{cases} a\hat{G}_0 + b\hat{\sigma}\hat{G}_0 - c\hat{\sigma}\hat{G}_1^* = \varepsilon\hat{\sigma} \\ a\hat{G}_1 + b\hat{\sigma}\hat{G}_1 - c\hat{\sigma}\hat{G}_0^* = 0. \end{cases} \quad (43)$$

The solution of these equations is obtained as

$$\hat{G}_0 = \frac{\varepsilon}{d^2 - e^2} ((db^* - ea^*)\hat{I} + (da^* - eb^*)\hat{\sigma}), \quad \hat{G}_1 = \frac{\varepsilon c}{d^2 - e^2} (d\hat{I} - e\hat{\sigma}), \quad (44)$$

where $d = aa^* + bb^* - cc^*$ and $e = ab^* + ba^*$.

The further calculation of the spectral variance using

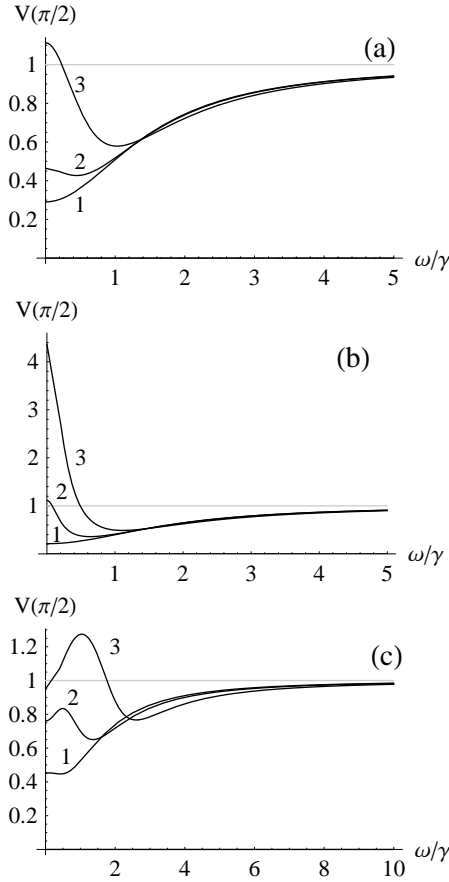


FIG. 2: The spectral variances versus ω/γ . The parameters are: a) $\Delta/\gamma = 0$, $\chi/\gamma = 0, 0.2, 0.5$ (curve 1, 2, 3), $\varepsilon/\varepsilon_{th} = 0.5$. b) $\Delta/\gamma = 0$, $\chi/\gamma = 0, 0.2, 0.5$ (curve 1, 2, 3), $\varepsilon/\varepsilon_{th} = 0.8$. c) $\Delta/\gamma = 0.2, 0.5, 1$ (curve 1, 2, 3), $\chi/\gamma = 0.05$, $\varepsilon/\varepsilon_{th} = 0.5$

the P -representation is standard. A detailed description of the method can be found in [29]. We include also the output coupler transmissivity 2γ for cavity output fields so that the frequency spectrum of the squeezing variance $V(\theta, \omega)$ corresponding to the integral variance in the formula (23) is calculated to be

$$V(\theta, \omega) = 1 + S^{(1)}(\theta, \omega - \Delta) + S^{(1)}(\theta, -\omega - \Delta), \quad (45)$$

where

$$S^{(1)} = \frac{\gamma}{\pi} (G_{1,11}(\omega) + G_{1,22}(-\omega) + 2\text{Re}[e^{2i\theta} G_{0,21}^*(\omega)]), \quad (46)$$

and $G_{0,21}^*$, $G_{1,11}$, $G_{1,22}$ are the elements of the matrices \hat{G}_0^* , \hat{G}_1 . The variance $V(\theta, \omega)$ of the output quadrature amplitudes is symmetric in frequency around $\omega = 0$, and equal to unity in the vacuum or coherent signal case, and can only reach its minimum value at the definite frequency. For the case of ordinary NOPO, if $\chi = 0$, this result is coincided with the analogous one well known result [29]. Typical results for self-locked NOPO are shown in Figs.2, for the various parameters χ and for $\theta = \pi/2$

chosen to minimize the noise level $V(\theta, \omega)$. The standard quantum limit is shown in grey. We see that the minimal variance spectra remain less than unity for all frequencies only for the cases of small values of the parameter χ .

Figures 2(a), 2(b) plot the squeezing spectra in the absence of the detunings ($\Delta_1 = \Delta_2 = 0$) for two values of the pump field: $\varepsilon = 0.5\varepsilon_{th}$ (Fig. 2(a)) and $\varepsilon = 0.8\varepsilon_{th}$ (Fig. 2(b)), and for the various values of the parameter χ : $\chi = 0$ (curves 1), $\chi = 0.2$ (curves 2), $\chi = 0.5$ (curves 3). The excellent squeezing spectra centered at $\omega = 0$, occurs near threshold for $\chi = 0$ that is the case of an ordinary NOPO [29]. We point out that there is a corresponding decrease in the squeezing as the parameter χ increases. Note, that for phase-locked NOPO the minima of spectra are shifted from the zero frequency (see Fig.2 (a), (b) curves 2 and 3) even in the case of zero detunings. Figures 2(c) plot squeezing spectra for the case of nonzero detuning. These results for $\chi = 0$ are in a good agreement with the results obtained for an ordinary NOPO [29].

IV. SELF-PULSING REGIME AND ENTANGLEMENT

Now let us pay our attention to the case of classically nonstationary regime of generation, when the inequality (7) does not valid. In this regime analytical treatment of semiclassical and quantum equations is complicated, therefore, we simulate Eqs. (3) and (4) on one side, and use the QSD method to obtain numerical solution of Eq. (1) on the other side. We find that the semiclassical solution of the equation (3), (4), without the noise terms and for $\beta_i = \alpha_i^*$, exhibits the self-pulsing instability: the photon number of intracavity modes oscillates periodically. We demonstrate this oscillations in Fig.3 (a) (curve 1) for one of the modes. It appears that self-pulsing exists in the whole range of violation of the inequality (7), and does not depend on the what parameter is changed to violate it. We present also the phase space trajectory of the semiclassical solution in Fig.4 (a), which has the form of squeezed circle. The individual quantum trajectory of the single mode of self-locked NOPO for the same parameters as in Fig.3(a) is presented in Fig.3 (b) as a quantum mechanical calculation on the base of QSD method. It is seen that trajectory repeats the oscillations of semiclassical solution.

Due to the quantum noise, the averaged over ensemble of quantum trajectories result does not exhibit oscillations (see Fig.3(a), curve 2). The analogous situation takes place for various quantum systems, particularly, also for chaotic systems, where quantum trajectory reflects the chaotic nonstationary behavior of the semiclassical trajectory, but ensemble averaged results has stationary solution (see [30]). It has been shown in Ref.[30], that quantum chaos manifests itself, particularly, in the Wigner function: it repeats the shape of Poincaré section of semiclassical counterpart. So, it is expected that

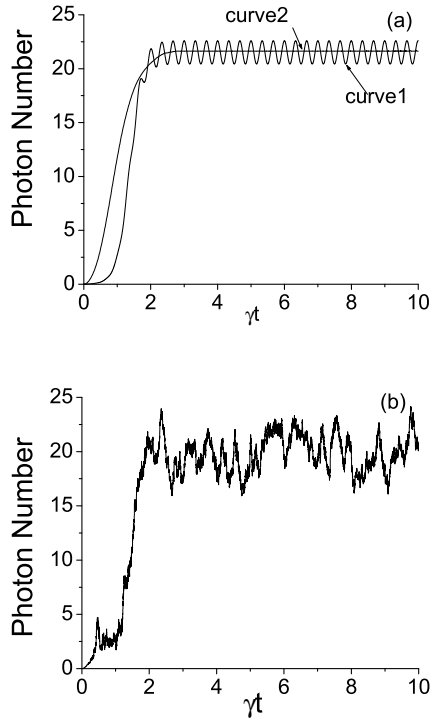


FIG. 3: Time dependence of the photon number: (a) classical trajectory (curve 1) and quantum ensemble averaged result (curve 2), and (b) quantum trajectory of self-phase locked NOPO in the regime of self-pulsing. The parameters are: $\lambda/\gamma = 0.1$, $\Delta_1/\gamma = 10$, $\Delta_2/\gamma = -5$, $\chi/\gamma = 0.5$, $\varepsilon/\gamma = 4$. Curve 2 on the Fig.3(a) involve averaging over 12000 quantum trajectories.

the features of the phase trajectory (Fig.4(a)) will be reflected in the Wigner function also for the system under consideration. Indeed, as it is seen from Fig.4(a) and Fig.4(b), where the contour-plots of the Wigner function is depicted, it is really the case. Note, that the result of Fig.4(b) is obtained in the framework of QSD method. Thus, we can conclude that the photon number of self locked NOPO is stationary in the full quantum treatment, even when the semiclassical counterpart in non-stationary. However, self pulsing manifests itself in the Wigner function, which is rather different from the case of stationary stable regime (see Fig.1 and Fig. 4(b)).

It is also interesting to compare the result of Fig.4(b), which has a slightly squeezed circle form, with the Wigner function of ordinary NOPO ($\chi = 0$), presented in the paper [31], which has the form of circle. This form is the manifestation of phase diffusion. Indeed, the phase in phase space is proportional to the angle between the Ox axis and radius-vector of the point. As the Wigner function distributed uniformly to all directions, this mean that the phase operator is entirely uncertain, which is equivalent to phase diffusion. For the case of self locked NOPO the distribution is almost uniform to all direc-

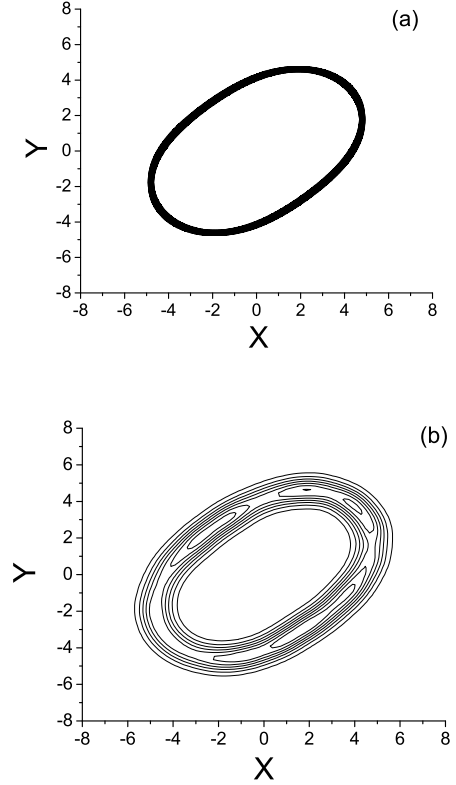


FIG. 4: Semiclassical phase space trajectory (a) and contour-plots of the Wigner function (b) of the self-phase locked NOPO in the regime of self-pulsing. The parameters are: $\lambda/\gamma = 0.1$, $\Delta_1/\gamma = 0.1$, $\Delta_2/\gamma = -0.1$, $\chi/\gamma = 0.5$, $\varepsilon/\gamma = 3$. Averaging in Fig.4 (b) is over 3000 quantum trajectories.

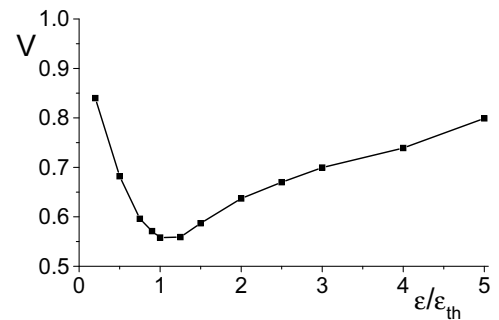


FIG. 5: Entanglement versus pump amplitude. The parameters are: $\lambda/\gamma = 0.1$, $\Delta_1/\gamma = 10$, $\Delta_2/\gamma = -10$, $\chi/\gamma = 0.1$.

tions, which means that the modified phase diffusion also takes place here. The squeezed form of the circle is due to oscillation of photon number.

Analogous results, but for triply resonant cavity have been obtained numerically by Groß and Boller [7] in the framework of semiclassical treatment: the phases of the modes of self locked NOPO in the self-pulsing regime diverge. Moreover, the rate of the phase diffusion is the same as for an ordinary NOPO. It is interesting that

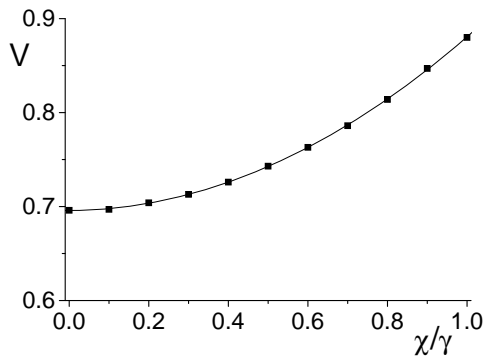


FIG. 6: Entanglement versus wave plate parameter χ/γ (squares - numerical result, line - quadratic fit). The parameters are: $\lambda/\gamma = 0.1$, $\Delta_1/\gamma = 10$, $\Delta_2/\gamma = -10$, $\varepsilon/\gamma = 3$.

the semiclassical phase space trajectory in Fig.4(a) has similar form as in the case of triply resonant NOPO.

Another interesting question, which arises from studying nonstationary regime is: whether entanglement takes place in this regime or not? And, if yes, does it reach the same extent or perhaps is higher with respect to the stationary regime. Nevertheless, we understand that it will not have the desirable property of self-phase locking. For answering these questions, we consider the dependence of dispersion V (23) on both driving field amplitude in units of its threshold value $\varepsilon/\varepsilon_{th}$, and quarter wave plate parameter χ . At first, we numerically estimate that $\varepsilon_{th}/\gamma = 1$ for the parameters used, and then we consider the dependence of V on $\varepsilon/\varepsilon_{th}$. Then, we perform calculations and the results are presented in Fig.5. It is seen, that the variance drops down near the threshold and achieve the minimal value $V = 0.55$ at the threshold. Then, it increases by increasing the driving amplitude. This behavior is similar to that for the case of stable operational regime of self-phase locked NOPO [12].

We also present here the dependence of V versus wave plate parameter χ in Fig. 6, where the decreasing of entanglement degree by increasing of χ is clearly evident. This means, that while, as mentioned in Sec.II, the local-

ization of phases of the modes is improved with increasing of χ , the entanglement is worsening.

It is remarkable, that this dependence is exactly quadratic: we fit the numerical results (squares in Fig.6) with quadratic curve and find excellent coincidence (solid curve in Fig.6).

V. CONCLUSION

In conclusion, we have studied in the full quantum mechanical manner the properties of the light beams generated in self-phase locked NOPO. We have continued the recent investigations of the quantum aspects of this device in the steady state, stable regime of generation [12] in one side and also we have considered the specific signatures of the NOPO containing a birefringent element in the unstable regime of generation on the other side. In this latter case, the system exhibits the both self-pulsing temporal behavior and a new type of phase diffusion as it has been demonstrated by numerical simulations on the framework of the Wigner function. The entanglement does occur in this regime too, but suffer some worsening: it decreases quadratically when wave plate parameter increases linearly. Considering the stable regime of generation, we conclude that significant noise reduction in intensity difference variance as well as in the quadrature squeezing spectra are possible for small parameters $\chi/\gamma \ll 1$. A detailed analysis of two-mode squeezing spectra for self-phase locked NOPO below threshold has been given, using a P-presentation in applications to the experiment recently performed [14]. We have also analyzed photon-number correlations in the presence of phase localization which has also been studied on framework of the Wigner function.

VI. ACKNOWLEDGEMENTS

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